UDC 532.5

ON WEBER EQUATIONS

PMM Vol. 38, № 5, 1974, pp. 829-836

B. I. MUKOSEEV
(Arzamas)

(Received November 16, 1973)

The equations of motion of a continuous medium for a barotropic fluid were derived by Weber [1] in Lagrange variables, and Serrin [2] extended these to the isentropic flow. It is shown here that the left-hand part of Weber equation represents "in the small" the total velocity (an analog of the Cauchy-Helmholtz theorem). These equations are extended to flows of a viscous compressible fluid, and an attempt is made to specify with the use of the Clebsch theorem [2] the surface forces in terms of gradients of potential functions. This is shown to be feasible in a number of practically important cases.

1. Derivation of Weber equations. We write the equations of motion of a continuous medium in the form [2]

$$\rho \mathbf{u}^{\cdot} = \mathbf{F} + \mathbf{P} \tag{1.1}$$

where ρ is the mass density, $\mathbf{u}$ is the velocity vector of a fluid particle, and $\mathbf{F}$ and $\mathbf{P}$ are the vectors of mass and surface forces, respectively.

We use the following notation: x_k and a_k denote Euler and Lagrange variables, respectively; a dot denotes a total derivative with respect to time t; $\partial u_j / \partial x_k = u_{j,k}$ and $\partial u_j / \partial a_k = u_{j,\bar{k}}$ denote partial derivatives with respect to Euler and Lagrange variables, respectively; conventional operators of vector analysis written in lower case letters relate to Euler variables, while those expressed in capitals relate to Lagrange variables; subscripts j and k run through 1, 2, 3: δ_{jk} denotes the Kroneker delta; μ is the coefficient of shear viscosity; λ is the coefficient of second viscosity; u_j^p is the translational velocity of a fluid particle; $\omega_{jk} = {}^1/_2(u_{j,\bar{k}} - u_{k,\bar{j}})$ is the angular velocity of a fluid particle; $\varepsilon_{jk} = {}^1/_2\,(u_{j,\bar{k}} + u_{k,\bar{j}})$ is the relative strain of a fluid particle; $v = \rho^{-1}$ is the specific volume; σ is the stress tensor; $I = E + p / \rho$ is the enthalpy; E is the internal energy; T is the temperature; $\varkappa$ is the Stefan-Boltzmann constant; $\mathbf{r} = \mathbf{i}_1 x_1 + \mathbf{i}_2 x_2 + \mathbf{i}_3 x_3$ is the position vector of the fluid particle trajectory; $\mathbf{i}_k$ are unit vectors; S is the entropy; $c = (\gamma p v)^{1/2}$ is the speed of sound, and γ is the ratio of specific heats.

We denote the formulas for the transformation of representation in Euler variables to that in Lagrange variables as follows: the Jacobian

$$\frac{\partial (f, x_2, x_3)}{\partial (a_1, a_2, a_3)} = [f, x_2, x_3] \quad \left(\text{for} \quad f = x_1 \quad J = \frac{\partial (x_1, x_2, x_3)}{\partial (a_1, a_2, a_3)}\right)$$

$$\operatorname{grad} f = J^{-1} (\mathbf{i}_1 [f, x_2, x_3] + \mathbf{i}_2 [x_1, f, x_3] + \mathbf{i}_3 [x_1, x_2, f])$$

$$\operatorname{div} f = J^{-1} ([f, x_2, x_3] + [x_1, f, x_3] + [x_1, x_2, f]) \quad \text{etc.}$$

If we denote the potential of the mass force vector by Ω, then $\mathbf{F} = -\operatorname{grad} \Omega$. Multiplying the two sides of Eq. (1.1) by $\operatorname{grad} \mathbf{r}$, we have

$$\frac{d}{dt}(\mathbf{u}\cdot\operatorname{grad}\mathbf{r}) = \operatorname{grad}\left(\frac{u^2}{2} - \Omega\right) + v\mathbf{P}\operatorname{grad}\mathbf{r} \tag{1.2}$$

$$(\mathbf{u}\cdot\operatorname{grad}\mathbf{r} = \mathbf{i}_1(\mathbf{r}_{,\bar 1}\,\mathbf{u}) + \mathbf{i}_2(\mathbf{r}_{,\bar 2}\,\mathbf{u}) + \mathbf{i}_3(\mathbf{r}_{,\bar 3}\mathbf{u}) = \mathbf{i}_k(\mathbf{r}_{,\bar k}\mathbf{u}))$$

Integration of Eq. (1.2) with respect to time from t_0 to t yields the Weber equation of motion

$$U - U_0 = \operatorname{grad}\psi + A \tag{1.3}$$

where

$$\psi = \int_{t_0}^{t}\left(\frac{u^2}{2} - \Omega\right)d\tau, \quad \mathbf{A} = \int_{t_0}^{t} v\mathbf{P}\operatorname{grad}\mathbf{r}\,d\tau, \quad \mathbf{U} = \mathbf{u}\operatorname{grad}\mathbf{r}$$

and the subscript zero indicates that the related quantity is taken at the instant of time t_0. In projections on the axes of rectangular coordinates $OX_1X_2X_3$ Eq. (1.3) assumes the form

$$\sum_{j=1}^{3}(u_j x_{j,\bar k} - u_{0j}x_{0j,\bar k}) = \psi_{,\bar k} + A_k \tag{1.4}$$

It can be shown that vector $\mathbf{U}$ is the algebraic sum of three vectors: the vector of translational velocity $\mathbf{u}^p$, the vector of deformation rate, and the vector of rotational velocity (an analog of the Cauchy-Helmholtz theorem [4]). To prove this statement we use the Kirchhoff formula [5]

$$x_{j,\bar k} = \delta_{jk} + u_{j,\bar k}\delta t$$

Substituting this relationship into the left-hand part of (1.3) and carrying out the computations expounded in [4], we obtain the following expression for the left-hand part of Eq. (1.3):

$$U_j = u_j^p + \varepsilon_{jk}\delta x_k - \omega_{jk}\delta x_k$$

in which all terms are the same as in the Cauchy-Helmholtz formulas [4], except of the opposite sign at $\omega_{jk}\delta x_k$, which is due to the Lagrange form of presentation. This proves that vector $\mathbf{U}$ represents the total velocity.

For reference we present the formulas which relate velocities $\mathbf{u}$ and $\mathbf{U}$ (the derivative of U_k is not taken):

$$u_1 = J^{-1}[U_1, x_2, x_3], \quad u_2 = J^{-1}[x_1, U_2, x_3], \quad u_3 - J^{-1}[x_1, x_2, U_3]$$

2. Some particular forms of equation (1.3). Equations of the form (1.3) are complex. To simplify these we begin with the formula for the surface force $\mathbf{P}$ which we determine as follows. Using the Clebsch theorem [3] which states that any vector may be presented in the form

$$\mathbf{P} = \operatorname{grad}\alpha + \beta\operatorname{grad}\chi \tag{2.1}$$

where α, β and χ are as yet unknown potential functions, we represent $\mathbf{A}$ in the form

$$\mathbf{A} = \int_{t_0}^{t} v(\operatorname{grad}\alpha + \beta\operatorname{grad}\chi)\,d\tau$$

To determine α, β and χ we consider specific mathematical models.

Perfect fluid. Let the surface forces be expressed in terms of stress tensors [2], i.e. $\mathbf{P} = \operatorname{div}\sigma$. Using the additivity hypothesis [6] it is possible to decompose tensor σ into reversible and irreversible parts

$$\sigma = \sigma_{jk}^{(r)} + \sigma_{jk}^{(i)} \tag{2.2}$$

The rheological equation for a perfect incompressible fluid is of the form [6]

$$\sigma_{jk}^{(r)} = -p, \qquad \sigma_{jk}^{(i)} = 0$$

Setting $\alpha = -p$ and $\beta = \chi = 0$, after some simple computations we reduce the equation of motion (1.3) to the form

$$\mathbf{U} - \mathbf{U}_0 = \operatorname{grad}\varphi, \quad \varphi^{\cdot} = \frac{u^2}{2} - \frac{p}{\rho} - \Omega \tag{2.3}$$

This implies that the difference $\mathbf{U} - \mathbf{U}_0$ of total velocity vectors is a potential vector. Note that there is an example of vortex fluid flow (Gerstner's problem [2]) in which the fluid particle velocity $\mathbf{u}$ is not potential. For a barotropic compressible fluid the equation of motion (adiabatic flow) is of the form

$$\mathbf{U} - \mathbf{U}_0 = \operatorname{grad}\varphi, \qquad \varphi^{\cdot} = \frac{u^2}{2} - I - \Omega \tag{2.4}$$

These equations were derived by Weber [1]. For an isentropic flow ($S^{\cdot} = 0$) we have [2]

$$\mathbf{U} - \mathbf{U}_0 = \operatorname{grad}\varphi + \eta \operatorname{grad} S \tag{2.5}$$

$$\varphi^{\cdot} = \frac{u^2}{2} - I - \Omega, \qquad \eta^{\cdot} = T$$

Viscous fluid. Assuming that for a viscous incompressible fluid $\rho = \text{const}$, $\alpha = -p$, $\beta = \mu = \text{const}$, $\nu = \mu\rho^{-1}$ and $\operatorname{rot}\omega = \operatorname{grad}\chi$, we have

$$\mathbf{U} - \mathbf{U}_0 = \operatorname{grad}\varphi, \qquad \varphi^{\cdot} = \frac{u^2}{2} - \frac{p}{\rho} - \Omega - \nu\chi \tag{2.6}$$

These equations are valid for the Couette, Poiseuille, Beltrami, Stokes and generally slow flows.

If we set $\alpha = -p + \lambda\theta^{\cdot}$, $\beta = \chi = 0$ and $\theta^{\cdot} = \operatorname{div} u$ ($\theta = \ln J$), we obtain the equations of motion for a compressible fluid in which second viscosity is taken into account

$$\mathbf{U} - \mathbf{U}_0 = \operatorname{grad}\varphi - \int_{t_0}^{t} \alpha \operatorname{grad} v\, d\tau, \quad \varphi^{\cdot} = \frac{u^2}{2} - \Omega - v(p - \lambda\theta^{\cdot}) \tag{2.7}$$

The system of equations (2.7) can be extended to an isotropically radiating gas. In that case it is necessary to set $\alpha = -p + \lambda\theta^{\cdot} - \varkappa T^4$ and $\beta = \chi = 0$. We then have

$$\mathbf{U} - \mathbf{U}_0 = \operatorname{grad}\varphi - \int_{t_0}^{t} \alpha \operatorname{grad} v\, d\tau, \quad \varphi^{\cdot} = \frac{u^2}{2} - \Omega - v(p - \lambda\theta^{\cdot} + \varkappa T^4) \tag{2.8}$$

If we set $\beta = \mu = \text{const}$, $\operatorname{rot}\omega = \operatorname{grad}\chi$ and $\alpha = -p + \lambda\theta^{\cdot} - \varkappa T^4$, the equations assume the form

$$\mathbf{U} - \mathbf{U}_0 = \operatorname{grad}\varphi - \int_{t_0}^{t} (\alpha + \mu\chi) \operatorname{grad} v\, d\tau \tag{2.9}$$

$$\varphi^{\cdot} = u^2/2 - \Omega - v(p - \lambda\theta^{\cdot} + \varkappa T^4 + \mu\chi)$$

The shear viscosity is partially accounted for in this system of equations. The Couette flow with radiation may be considered as belonging to these flows.

Let us pass to the derivation of equations of motion for a viscous compressible fluid in which the reversible part of tensor $\sigma_{jk}^{(r)}$ for a Newtonian fluid is simply equal to pressure p, i.e. $\sigma_{jk}^{(r)} = -p$, while its irreversible part is

$$\sigma_{jk}^{(i)} = \lambda\theta^{\cdot}\delta_{jk} + \mu(u_{j,k} + u_{k,j})$$

With the use of the Clebsch theorem [3] it is possible to represent the total velocity vector in the form

$$\mathbf{U} = \operatorname{grad} h + f \operatorname{grad} g$$

and obtain the equation of motion of the form

$$\mathbf{U} - \mathbf{U}_0 = \operatorname{grad} \psi + \int_{t_0}^{t} v \{\operatorname{grad} [-p + (\lambda + \mu) \theta^{\cdot}] + \theta^{\cdot} \operatorname{grad} \mu + \mathbf{B}\} d\tau \qquad (2.10)$$

$$\mathbf{B} = \mathbf{i}_k [\operatorname{div} (\mu \operatorname{grad} \mathrm{u}_j)] \mathrm{x}_{j\bar{k}}$$

From the physical point of view the term B in Eq. (2.10) defines the transfer of momentum. It can be expressed in terms of Lagrange variables and thus yield particular cases such as: incompressible viscous fluid, perfect fluid, etc.

3. On the solution of particular problems. Stationary flow. Let us for simplicity consider a stabilized Couette flow. Boundary conditions are

$$u_1 = 0, \quad u_2 = 0 \qquad (x_2 = 0, \ x_2 = l)$$

where l is the distance between plates. We seek a solution of the form

$$x_1 = a_1 + \xi (a_1, a_2, t), \quad x_2 = a_2, \quad p = Aa_1$$

where A is a specified constant. From the continuity equation we have $x_{1,\bar{1}} = 1$, hence $\xi_{\bar{1}} = 0$.

Since the process is stabilized, $\mathbf{U} - \mathbf{U}_0 = 0$ and the system of the Navier-Stokes equations in terms of Lagrange variables is

$$\dot{\xi}_{,\bar{2}\bar{2}} = A\mu^{-1}$$

The solution with allowance for boundary conditions is of the form

$$x_1 = a_1 + \frac{Al^2}{2\mu}\left[\left(\frac{a_2}{l}\right)^2 - \frac{a_2}{l}\right], \quad x_2 = a_2, \quad p = Aa_1$$

This example shows that in the stationary case the equations are considerably simplified.

Nonstationary flow. So far we have considered a simplified surface force **P**. Let us consider now the inertial force. It was shown above that the total velocity can be decomposed into translational and rotational velocities and the rate of deformation. Using the principle of additivity, it is actually possible to separate the translational velocity from the total velocity by representing the trajectory of fluid particles in the form

$$x_k = a_k + \xi_k(a_j, t)$$

The total velocity is then

$$U_j = u_j + \sum_{k=1}^{3} u_k \xi_{k,\bar{j}} = u_j + \frac{1}{2}(u^2)_{,\bar{j}}\, dt$$

since the second term is small in comparison with velocity u_j, it can be neglected in certain cases.

As an example let us consider Gerstner waves [2]. This problem is interesting since it has an exact analytic solution. Substituting in Eq. (2.3) **u** for **U** and integrating the obtained equation once more with respect to t, we obtain

$$\xi - u_0 t = \operatorname{grad} \varphi, \qquad \varphi^{\cdot\cdot} = \frac{u^2}{2} - \frac{p}{\rho} - \Omega$$

Substituting Gerstner's solution [2]

$$x_1 = a_1 + m^{-1}\exp(ma_2)\sin m(a_1 + ct)$$
$$x_2 = a_2 - m^{-1}\exp(ma_2)\cos m(a_1 + ct)$$
$$\frac{p}{\rho} = \text{const} - ga_2 + \frac{c^2}{2}\exp(2ma_2)$$

into the obtained system of equations, we conclude that the latter is exactly satisfied for $c^2 = gm^{-1}$ (g is the acceleration of gravity, m^{-1} is the amplitude of oscillations, and c is the wave propagation velocity).

Let us consider a nonstationary Couette flow with the following initial and boundary conditions:

$$t = 0,\ u_1 = u_2 = 0, \qquad p = 0$$
$$x_2 = 0, \quad x_2 = l, \quad u_1 = u_2 = 0$$

In this case it is convenient to represent the solution in the form

$$x_1 = a_1 + \xi(a_1, a_2, t), \quad x_2 = a_2, \quad p = Aa_1$$

From the continuity equation we again have $x_{1,\bar{1}} = 1$, which yields the following equation:

$$\xi^{\cdot} = \nu\xi_{,\bar{2}\bar{2}} - f$$

The general solution of this problem is of the form

$$x_1 = a_1 + \frac{2}{l}\int_0^t\int_0^l\left\{\sum_{n=1}^{\infty}\exp[-\omega_n^2(t-\tau)]\sin\frac{n\pi a_2}{l}\sin\frac{n\pi b}{l}\right\}f(b,\tau)\,db\,d\tau$$

Let us consider the particular case of

$$f(b,\tau) = \frac{A_0}{\alpha\rho}[1 - \alpha\tau - \exp(-\alpha\tau)]$$

The solution for x_1 is

$$x_1 = a_1 - 4A_0l^4(\pi^5\mu\nu)^{-1}\sum_{n=1}^{\infty}(2n-1)^{-5}\left[\frac{\omega_n^2+\alpha}{\alpha} + \frac{\alpha}{\omega_n^2-\alpha}\times\right.$$
$$\left.\exp(-\omega_n^2 t) - \omega_n^2 t - \frac{\omega_n^4}{\alpha(\omega_n^2-\alpha)}\exp(-\alpha t)\right]\sin\frac{(2n-1)\pi a_2}{l}$$

$$(\omega_n^2 = (2n-1)^2\pi^2\nu l^{-2})$$

For the stationary mode the expression for maximum velocity $(x_2 = l/2)$ may be written as

$$u_{1\max} = A_0 l^2 / 8\mu$$

The Cauchy problem. Let the adiabatic flow of a barotropic fluid be defined by the following system of equations of motion, by the equation of continuity, and the equation of state: $\mathbf{U} - \mathbf{U}_0 = \text{grad}\,\varphi, \quad \varphi^{\cdot} = u^2/2 - I + \lambda v^{\cdot}, \quad u_k = x_k^{\cdot}$ (3.1)

$$\rho_0 J_0 = \rho J \tag{3.2}$$
$$I = I_0(\rho/\rho_0)^{\gamma-1} \tag{3.3}$$

with the following initial conditions $(t = t_0)$:

$$x_k = x_{0k}(a_j), \quad u_k = u_{0k}(a_j), \quad \rho = \rho_0, \quad I = I_0 \tag{3.4}$$

We shall assume that the unknown functions are continuous and have continuous unbounded derivatives with respect to specified variables. The problem thus stated is complex, however, by introducing new variables of the form $b_k = x_{0k}(a_j)$ it can be somewhat simplified.

The form of the system of equations of motion (3.1) remains unchanged, but all derivatives with respect to spatial variables will be expressed in terms of new variables and the equation of continuity assumes the form

$$\rho_0 = \rho J \tag{3.5}$$

where the Jacobian J is expressed in terms of new variables. This can be proved by direct substitution.

Prior to passing to the derivation of solution, we shall carry out a preliminary analysis. Representing density and specific volume in the form $\rho = \rho_0(1 + \rho')$ and $v = v_0(1 + v')$, we write $v' = (1 + \rho')^{-1} - 1, \quad v' = J - 1$

From these expressions follows

$$\rho' = 0, \quad v' = 0, \quad J = 1, \quad \rho' \to \infty, \quad v' \to -1, \quad J \to 0$$

hence the specified quantities vary within the following limits:

$$0 \leqslant \rho' < \infty, \quad -1 < v' \leqslant 0, \quad 0 < J \leqslant 1$$

Taking this into consideration it is possible to expand the equation of state (3.3) into a binomial series

$$I = I_0 \left\{1 + \sum_{g=1}^{\infty} \left(-\frac{v'}{q!}\right) \prod_{n=1}^{q} (\gamma + n - 2)\right\} \tag{3.6}$$

We disregard in what follows the mass forces and consider the second viscosity coefficient as constant. After these preliminaries we can pass to the final statement of the Cauchy problem. Initial conditions (3.4) assume the form

$$x_k = b_k, \quad u_k = u_{0k}(b_j), \quad I = I_0(b_k), \quad v' = 0$$

We seek the solution of the form

$$x_k = \sum_{m=0}^{\infty} t^m x_k{}^m (b_k) \tag{3.7}$$

and of a similar form for other functions. After the substitution of series (3.7) into the altered system of Eqs. (3.1), (3.5) and (3.6) we obtain the following recurrent relationships:

the equation of motion

$$x_k{}^1 - u_{0k} + \sum_{m=1}^{\infty} t^m \left\{(m+1)\, x_k^{m+1} + \frac{1}{m}\left[I^{m-1}_{,\bar{k}} - \lambda v^m_{,\bar{k}} + \sum_{n=1}^{m} (n-1)\, n \sum_{j=1}^{3} x_j{}^n x_{j,\bar{k}}^{m-n+1}\right]\right\} = 0$$

the equation of continuity

$$\sum_{m=0}^{\infty} t^m \left(v^m - v_0 \sum_{n=0}^{m} \sum_{l=0}^{n} \delta_{ijk} x^l_{i,\bar{1}}\, x^{n-l}_{j,\bar{2}}\, x^{m-n}_{k,\bar{3}} \right) = 0$$

$$(v^\circ = v_0)$$

and the equation of state

$$I = I_0\Big\{1 + \sum_{q=1}^{\infty}\Big[(-1)^q (q!)^{-1}\Big[\prod_{n=1}^{q}(\gamma + n - 2)\Big]\Big(-1 + \sum_{m=0}^{\infty} t^m \sum_{l=0}^{m}\sum_{k=0}^{l} \delta_{ijs} x_{i,\bar{1}}^{k}\, x_{j,\bar{2}}^{l-k}\, x_{k,\bar{3}}^{m-l}\Big)q\Big]\Big\}$$

From these recurrent relationships we can obtain (by equating terms with equal powers of t) the expressions for coefficients of series (3.7). The analysis of the latter shows their convergence, since they are majorized by the convergent series

$$|M|\,[1\,/\,(1\cdot 2) + 1\,/\,(2\cdot 3) + \ \ldots\ ]$$

The radius of convergence of series (3.7) is determined by the interval $0 < t < |M^{-1}|$, where M is the maximum value of the m-th derivative of input data. The existence of solution of the Cauchy problem is thereby proved. Its uniqueness follows from the uniqueness of the specification of input data.

Thus Weber equations in some particular cases (stationary periodic flows) yield very simple equations which can be solved by conventional analytical methods.

REFERENCES

1. Lamb, H., Hydrodynamics (Russian translation). Gostekhizdat, Moscow-Leningrad, 1947.
2. Serrin, J., Mathematical Foundations of Classical Mechanics of Fluids (Russian translation). Izd. Inostr. Lit., Moscow, 1963.
3. Kochin, N. E., Vector Calculus and Introduction to Tensor Calculus (in Russian). ONTI, Leningrad-Moscow, 1937.
4. Kochin, N. E., Kibel', I. A. and Roze, N. V., Theoretical Hydromechanics, Fizmatgiz, Moscow, 1963.
5. Kirchhoff, G. R., Mechanics. Izd. Akad. Nauk SSSR, 1962.
6. Ziegler, H., Extremum Principles of Thermodynamics of Irreversible Processes and Mechanics of Continuous Medium. "Mir", Moscow, 1966.

Translated by J. J. D.

UDC 532.526

THE METHOD OF SUCCESSIVE APPROXIMATIONS IN PROBLEMS OF THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER

PMM Vol. 38, № 5, 1974, pp. 837-846

N. N. SHAKHOV and IU. D. SHEVELEV
(Moscow)

(Received February 7, 1974)

The analytical method of calculation of a three-dimensional boundary layer in a compressible fluid stream is considered. The method is based on the use of successive approximations and is similar to that used in the case of incompressible